Basically a matrix is in Echelon form if the zero entries make a "staircase" down the rows.

$$\begin{bmatrix} 0 & \# & \# & \# & \# & \# & \# & \# \\ 0 & 0 & 0 & \# & \# & \# & \# & \# \\ 0 & 0 & 0 & 0 & \# & \# & \# & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where #'s are any number and the circled ones are nonzero.

We say a matrix is in <u>reduced echelon form</u> if it is in echelon form, i.e. satisfies properties 1) - 3), and additionally

4) The leading entry in each nonzero row is 1.

5) Each leading 1 is the only nonzero entry in its column

If we take the matrix above, it would be in reduced echelon form if it looked like

$$\begin{bmatrix} 0 & 1 & \# & 0 & \# & \# & \# & 0 \\ 0 & 0 & 0 & 1 & 0 & \# & \# & 0 \\ 0 & 0 & 0 & 0 & 1 & \# & \# & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

still have "staircase" and now the leading entries have been scaled to 1 and the entries above and below them are 0.

Using elementary row operations, any nonzero matrix can be row reduced to echelon and reduced echelon form.

- A matrix can be row reduced to many matrices in echelon form
- A matrix can only be row reduced to one matrix in reduced row echelon form.

We say two matrices are row equivalent if one can be obtained from the other via row reduction.

So to say this remark more formally:

Theorem

Each matrix is row equivalent to a unique reduced echelon matrix.

Defn's

- A pivot position in a matrix A is the position in A corresponding to a ~~the~~ leading 1 in the reduced echelon form of A.
- A pivot column is a column of A containing a pivot position.

Example

Row reduce the matrix to reduced echelon form and identify the pivots.

$$A = \begin{bmatrix} 3 & 3 & 12 & 15 & 0 \\ 1 & 2 & 6 & 12 & -3 \\ 2 & 2 & 8 & 18 & -8 \end{bmatrix}$$

Solution:

Pivot position

since first column nonzero

$$\begin{bmatrix} \boxed{3} & 3 & 12 & 15 & 0 \\ 1 & 2 & 6 & 12 & -3 \\ 2 & 2 & 8 & 18 & -8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 2 & 6 & 12 & -3 \\ 3 & 3 & 12 & 15 & 0 \\ 2 & 2 & 8 & 18 & -8 \end{bmatrix}$$

pivot column

- Need to have leading 1 for reduced form
- Could have also multiplied row 1 by $1/3$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \begin{bmatrix} \boxed{1} & 2 & 6 & 12 & -3 \\ 0 & \boxed{-3} & -6 & -21 & 9 \\ 0 & -2 & -4 & -6 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} \boxed{1} & 2 & 6 & 12 & -3 \\ 0 & \boxed{1} & 2 & 7 & -3 \\ 0 & -2 & -4 & -6 & -2 \end{bmatrix}$$

new pivot since column 2 nonzero

Once you have a new pivot (leading 1) use it to clear entries above and below it

$$\xrightarrow[2R_2 + R_3 \to R_3]{-2R_2 + R_1 \to R_1} \begin{bmatrix} \boxed{1} & 0 & 2 & -2 & 3 \\ 0 & \boxed{1} & 2 & 7 & -3 \\ 0 & 0 & 0 & \boxed{8} & -8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} \boxed{1} & 0 & 2 & -2 & 3 \\ 0 & \boxed{1} & 2 & 7 & -3 \\ 0 & 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

$$\xrightarrow[2R_3 + R_1 \to R_1]{-7R_3 + R_2 \to R_2} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 1 \\ 0 & \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

Pivot columns

- Reduced Echelon Form
- Pivot positions in $\boxed{}$

Pivot positions in A

$$\begin{bmatrix} \boxed{3} & 3 & 12 & 15 & 0 \\ 1 & \boxed{2} & 6 & 12 & -3 \\ 2 & 2 & 8 & \boxed{18} & -8 \end{bmatrix}$$

We can solve systems of equations by row reducing the augmented matrix to reduced echelon form.

Recall the first example: (on 1/22)

Reduced Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 0 & 1 & 2 & a \end{array}\right] \xrightarrow{\text{Row Reduction}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.

Example

Is the system consistent or inconsistent?

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 = 5 \\ 2x_1 + 6x_2 + 3x_3 + 2x_4 = 13 \\ 3x_1 + 9x_2 + 4x_3 + 4x_4 = 20 \end{cases}$$

If consistent, describe the solution set.

Solution

pivot

$$\begin{bmatrix} \boxed{1} & 3 & 1 & 1 & | & 5 \\ 2 & 6 & 3 & 2 & | & 13 \\ 3 & 9 & 4 & 4 & | & 20 \end{bmatrix} \xrightarrow[-3R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \begin{bmatrix} \boxed{1} & 3 & 1 & 1 & | & 5 \\ 0 & 0 & \boxed{1} & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 5 \end{bmatrix}$$

$$\xrightarrow[-R_2 + R_3 \to R_3]{-R_2 + R_1 \to R_1} \begin{bmatrix} \boxed{1} & 3 & 0 & 1 & | & 2 \\ 0 & 0 & \boxed{1} & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

$$\xrightarrow{-R_3 + R_1 \to R_1} \begin{bmatrix} \boxed{1} & 3 & 0 & 0 & | & 0 \\ 0 & 0 & \boxed{1} & 0 & | & 3 \\ 0 & 0 & 0 & \boxed{1} & | & 2 \end{bmatrix}$$

Reduced echelon form

No pivots in rightmost column so $\boxed{\text{consistent}}$

What is the solution set?

$$\begin{cases} x_1 + 3x_2 = 0 \\ x_3 = 3 \\ x_4 = 2 \end{cases} \longrightarrow \begin{cases} x_1 = -3x_2 \\ x_2 = x_2 \\ x_3 = 3 \\ x_4 = 2 \end{cases}$$

x_2 becomes a free variable since it doesn't correspond to a pivot column. This means our solution set is

$$\{(-3x_2, x_2, 3, 2)\}$$

where x_2 can take any value. Hence there are infinitely many solutions. More on this is §1.5.